GRASP Laboratory
General Robotics & Autonomous Systems and Processes

Mechatronic Modeling and Design with Applications in Robotics

Frequency Domain Models

Impedance and Mobility Functions of Mechanical Elements

Element	Time-Domain Model	Impedance	Mobility (Generalized Impedance)
Mass m	$m\frac{dv}{dt} = f$	$Z_m = ms$	$M_m = \frac{1}{ms}$
Spring k	$\frac{df}{dt} = kv$	$Z_k = \frac{k}{s}$	$M_k = \frac{s}{k}$
Damper b	$f = bv$	$Z_b = b$	$M_b = \frac{1}{b}$

Note: Frequency domain is a special case of Laplace domain

Commonly, frequency domain is used when dealing with impedance approaches.

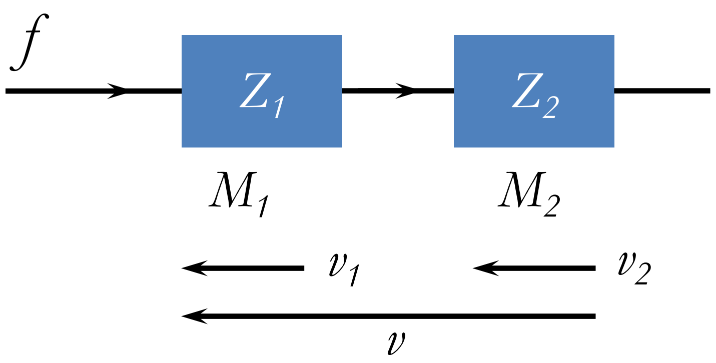
Impedance and Admittance Function of Electrical Elements

Element	Time-Domain Model	Impedance (*Z*)	Admittance (*W*)
Capacitor C	$c\frac{dv}{dt} = i$	$Z_c = \frac{1}{Cs}$	$W_c = Cs$
Inductor L	$L\frac{di}{dt} = v$	$Z_L = Ls$	$W_L = \frac{1}{Ls}$
Resistor R	$Ri = v$	$Z_R = R$	$W_R = \frac{1}{R}$

Note: Frequency domain is a special case of Laplace domain

Commonly, frequency domain is used when dealing with impedance approaches.

Series Connections

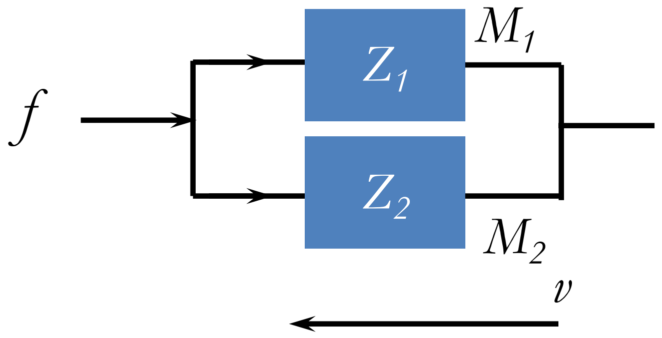


$$v = v_1 + v_2$$

$$\frac{v}{f} = \frac{v_1}{f} + \frac{v_2}{f}$$

$$M = M_1 + M_2 \qquad \frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2}$$

Parallel Connections



$$f = f_1 + f_2$$

$$\frac{f}{v} = \frac{f_1}{v} + \frac{f_2}{v}$$

$$Z = Z_1 + Z_2 \qquad \frac{1}{M} = \frac{1}{M_1} + \frac{1}{M_2}$$

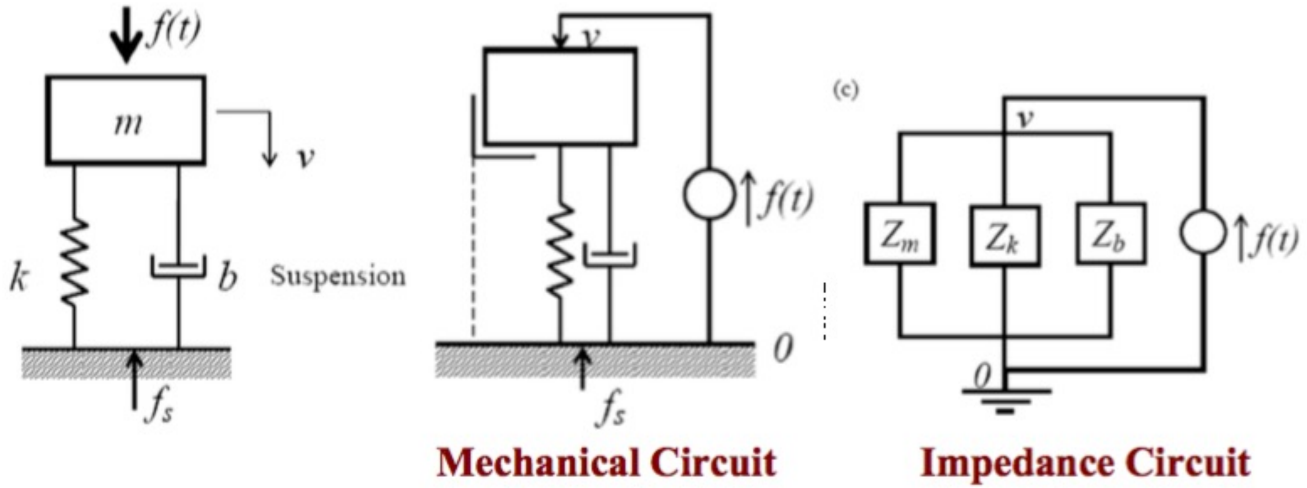
Interconnection Laws for Electrical Impedance and Admittance

Series Connections	Parallel Connections
$v = v_1 + v_2$	$i = i_1 + i_2$
$\frac{v}{i} = \frac{v_1}{i} + \frac{v_2}{i}$	$\frac{i}{v} = \frac{i_1}{v} + \frac{i_2}{v}$
$Z = Z_1 + Z_2$	$W = W_1 + W_2$
$\frac{1}{W} = \frac{1}{W_1} + \frac{1}{W_2}$	$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2}$

Note: Electrical Impedance and Mechanical Mobility are
"A-Type Transfer Functions" ← [Across Variable/Through Variable] ← Same Interconnection Law

Electrical Admittance and Mechanical Impedance are
"T-Type Transfer Functions" ← [Through Variable/Across Variable] ← Same Interconnection Law

Example 1: Ground-based Mechanical Oscillator

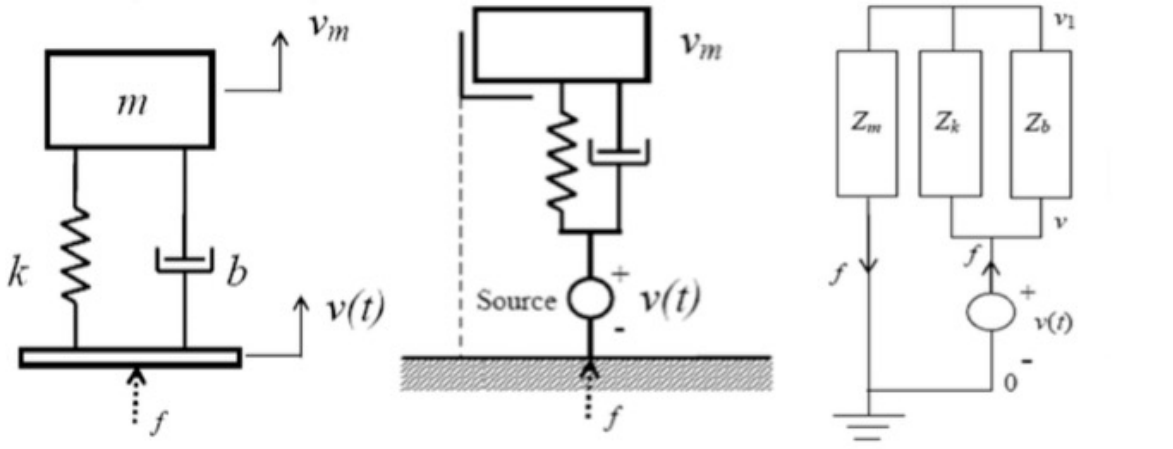


Three elements are connected in parallel ➔ Mechanical impedances add Overall Impedance Function $Z(s) = \frac{F(s)}{V(s)} = Z_m + Z_k + Z_b = ms + \frac{k}{s} + b = \frac{ms^2+bs+k}{s}$

Mobility Function $M(s) = \frac{V(s)}{F(s)} = \frac{s}{ms^2+bs+k}$

Note: Mobility (not mechanical impedance) is the natural transfer function for this system

Example 2: Oscillator with Support Motion



Mechanical Circuit **Impedance Circuit**

Spring and damper are connected in parallel: Their overall impedance =

$$Z_k + Z_b = \frac{1}{s} + b = Z_s$$

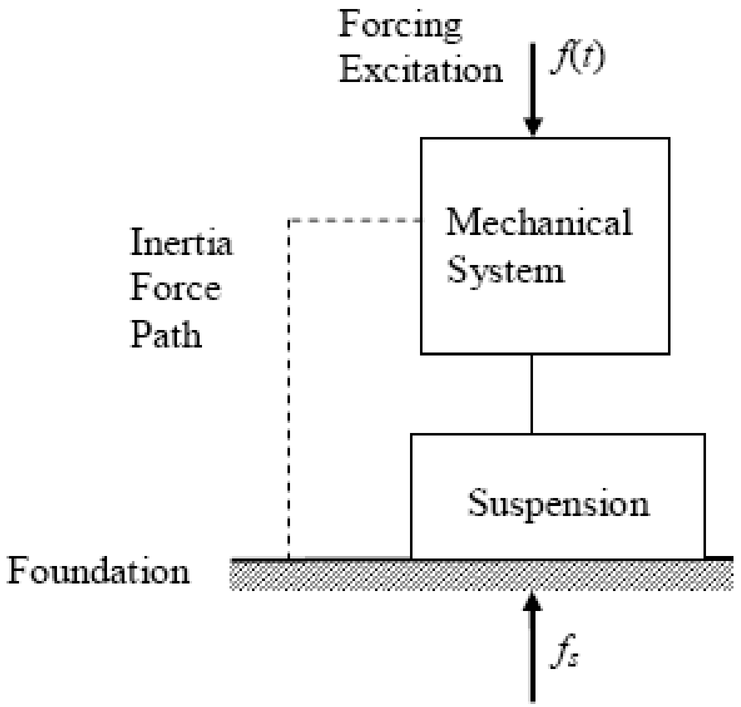
Mass is connected in series with this pair: Their overall mobility

$$\frac{V(s)}{F(s)} = M_m + \frac{1}{Z_s} = \frac{1}{ms} + \frac{1}{k/s + b} = \frac{ms^2 + bs + k}{ms(bs + k)}$$

Corresponding impedance $\frac{F(s)}{V(s)} = \frac{ms(bs+k)}{ms^2+bs+k}$; **Mobility of mass** $\frac{V_m(s)}{F(s)} = \frac{1}{ms}$ (s)

Transmissibility Functions

Force Transmissibility

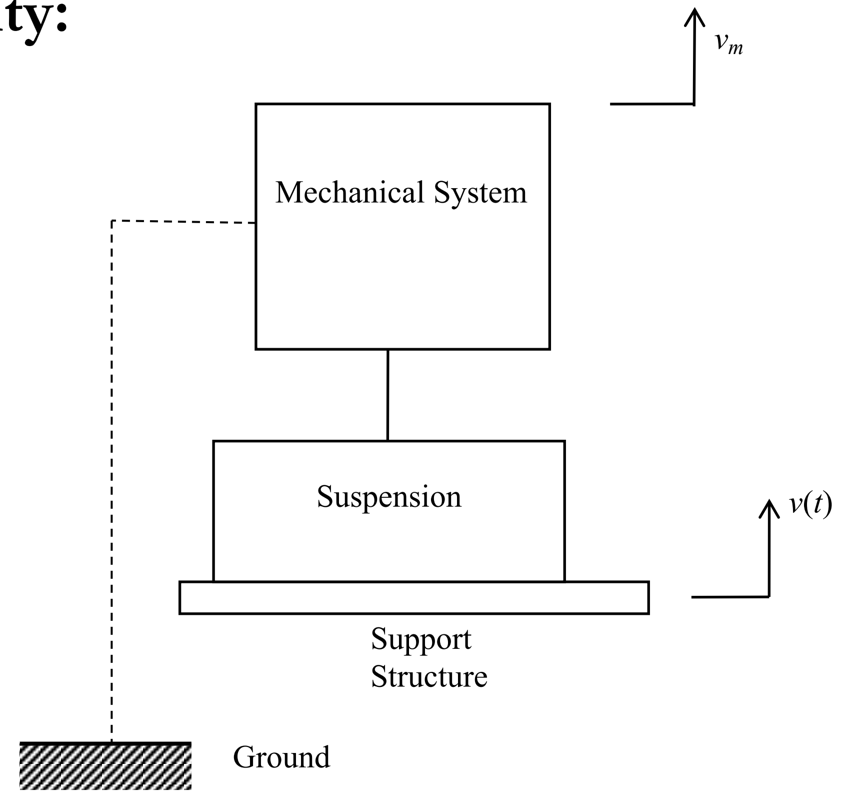


Think of a machine mounted on ground.

$$\text{Force Transmissibility } T_f = \frac{\text{Force Transmitted to Support } F_s}{\text{Applied Force } F}$$

***Note*: Inertia force path is not direct. Transmitted force f_s does not contain it.**

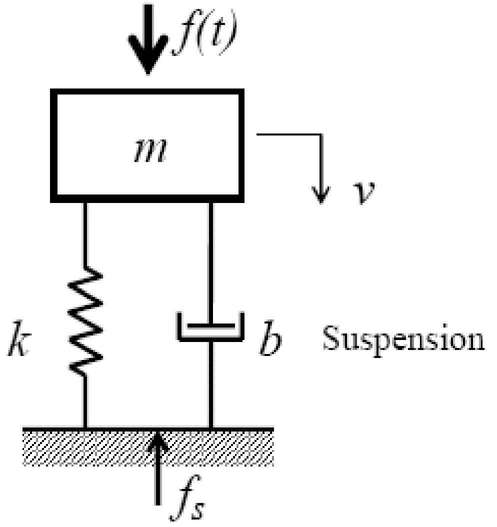
Motion Transmissibility:

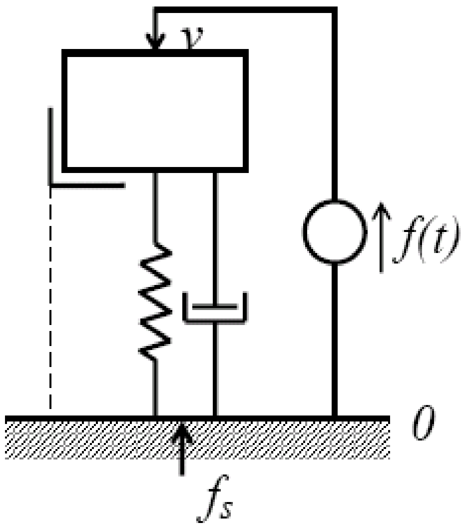


Think of a vehicle and its suspension system.

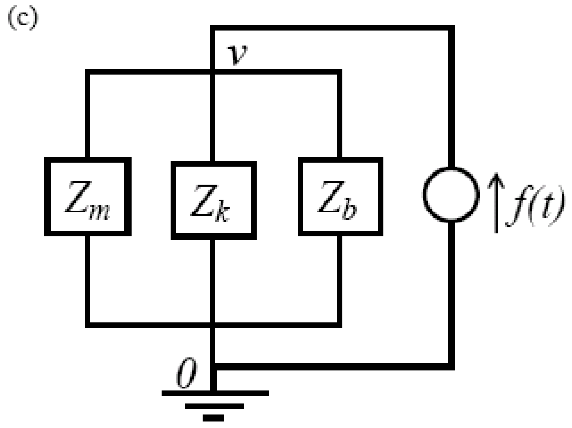
$$\textbf{Motion Transmissibility}\ T_m = \frac{\textbf{System Motion } V_m}{\textbf{Support Motion } V}$$

Example 1: Ground-based Mechanical Oscillator





Mechanical (Physical) Circuit



Impedance Circuit

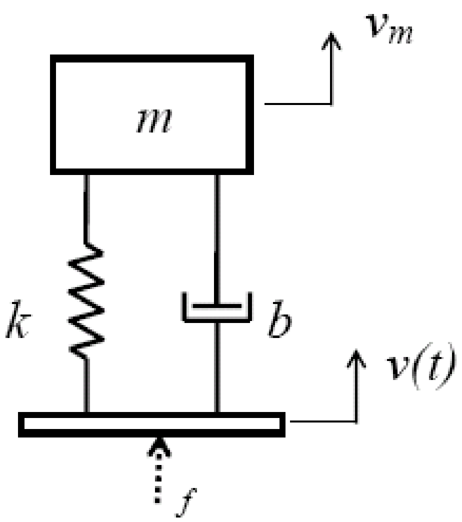
Parallel Connection➔ Common across variable; Through variables add

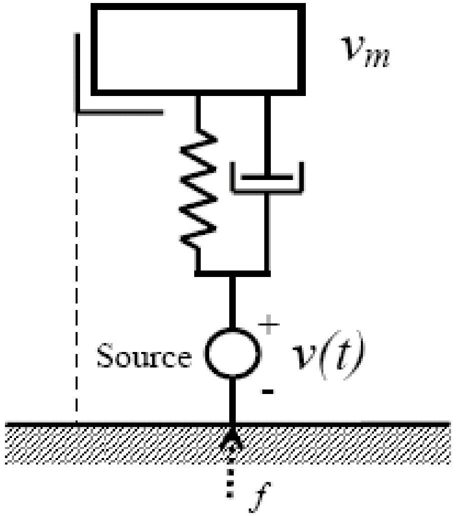
Force Transmissibility

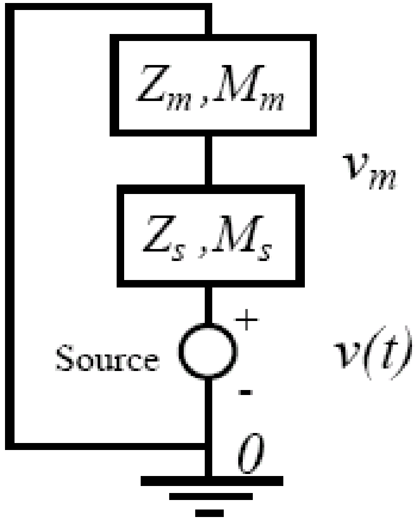
$$T_f = \frac{F_s}{F} = \frac{F_s / V}{F / V} = \frac{Z_b + Z_k}{Z_m + Z_b + Z_k} = \frac{Z_s}{Z_m + Z_s} = \frac{b + k/s}{ms + b + k/s} = \frac{bs + k}{ms^2 + bs + k}$$

➔ $T_f = \frac{Z_s}{Z_m + Z_s} = \frac{M_m}{M_s + M_m}$ ***Note*: Suspension Impedance** $Z_s = Z_b + Z_k = \frac{1}{M_s}$

Example 2: Oscillator with Support Motion







Think of a vehicle.

Mechanical (Physical) Circuit

Impedance Circuit

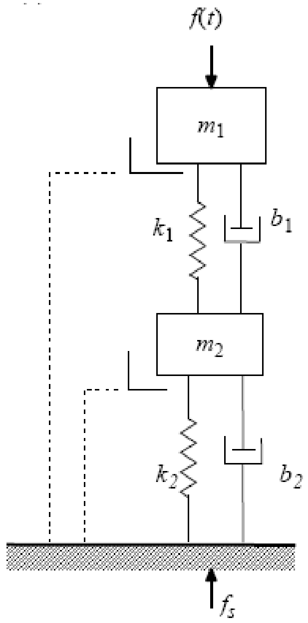
Motion Transmissibility

$$T_m = \frac{V_m}{V} = \frac{V_m / F}{V / F} = \frac{M_m}{M_m + M_s} = \frac{1}{1 + \frac{M_s}{M_m}} = \frac{1}{1 + \frac{Z_m}{Z_s}} = \frac{Z_s}{Z_m + Z_s}$$

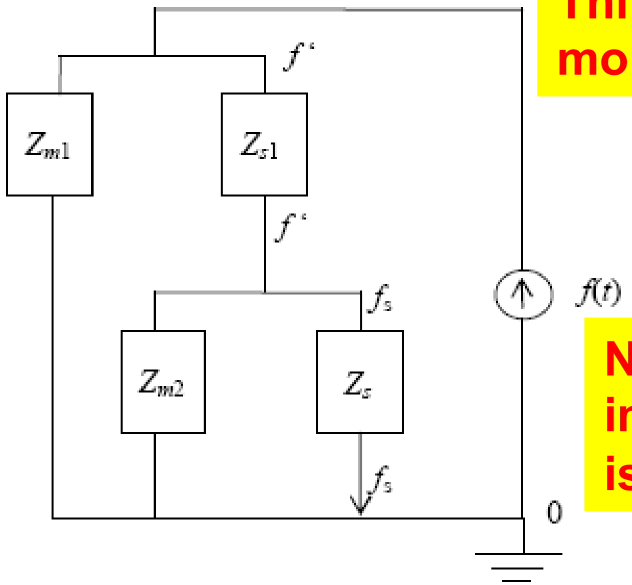
➔ T_f (Example 1) = T_m (Example 2)

➔ They are complementary (dual) systems for transmissibility

Example 3: Ground-based 2DOF Mechanical System



Mechanical System



Impedance Circuit

Think of mounted machine

Note how the impedance circuit is sketched

Mobility of right hand side branch $M = \frac{1}{Z_{s1}} + \frac{1}{Z_{m2} + Z_s}$

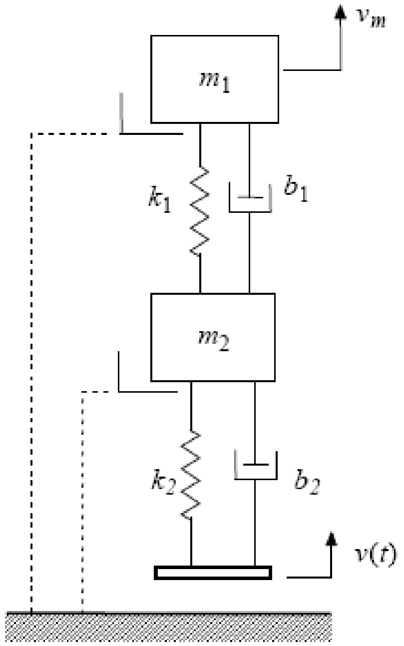
Force through branch $F' = \left[\frac{\frac{1}{M}}{Z_{m1} + \frac{1}{M}} \right] F = \left[\frac{1}{MZ_{m1} + 1} \right] F$

Force through Z_s is $F_s = \left[\frac{Z_s}{Z_{m2} + Z_s} \right] F'$

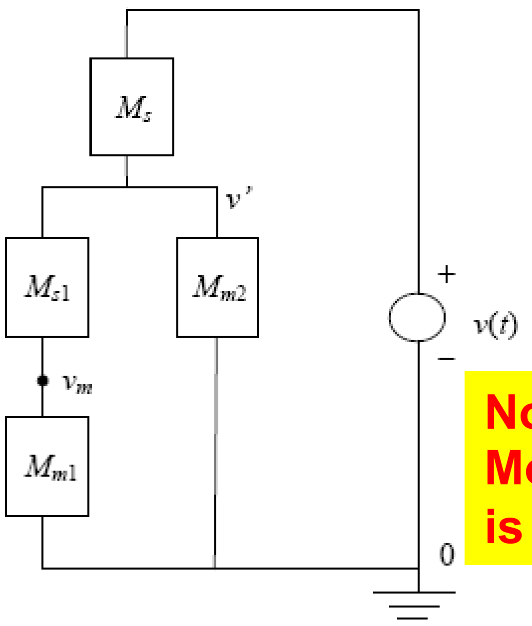
Force transmissibility $T_f = \frac{F_s}{F} = \left[\frac{1}{MZ_{m1} + 1} \right] \left[\frac{Z_s}{Z_{m2} + Z_s} \right]$

Force is divided in proportion to Impedance.

Example 4: 2DOF Mechanical System with Support Motion



Mechanical System



Impedance Circuit

Think of a vehicle

Note how the Mobility circuit is sketched

Impedance of bottom composite unit $Z = \frac{1}{M_{m2}} + \frac{1}{M_{s1} + M_{m1}}$

Velocity across this unit $V' = \left[\frac{\frac{1}{Z}}{M_s + \frac{1}{Z}}\right] V = \left[\frac{1}{M_s Z + 1}\right] V$

Velocity of mass m_1 is $V_m = \left[\frac{M_{m1}}{M_{s1} + M_{m1}}\right] V'$

Motion transmissibility $T_m = \frac{V_m}{V} = \left[\frac{1}{M_s Z + 1}\right]\left[\frac{M_{m1}}{M_{s1} + M_{m1}}\right]$

Velocity is divided in proportion to Mobility. Why?

The End!!